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The solution to the partition reconstruction problem

Maria Monks

290 Massachusetts Avenue, Cambridge, MA 02139, United States

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ABSTRACT

Given a partition λ of n , a k -minor of λ is a partition of $n - k$ whose Young diagram fits inside that of λ . We find an explicit function $g(n)$ such that any partition of n can be reconstructed from its set of k -minors if and only if $k \leq g(n)$. In particular, partitions of $n \geq k^2 + 2k$ are uniquely determined by their sets of k -minors. This result completely solves the partition reconstruction problem and also a special case of the character reconstruction problem for finite groups.

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1. Introduction

The problem of partition reconstruction can be stated as follows. For any positive integer k , define a k -minor of a partition λ of a positive integer $n > k$ to be a partition of $n - k$ whose Young diagram fits inside that of λ . It is natural to ask for which n and k we can uniquely determine any partition of n from its set of k -minors.

In this paper, we demonstrate that partitions of any positive integer $n \geq 2$ other than 5, 12, 21, and 32 can be reconstructed from their k -minors if and only if $k \leq \min_{0 \leq t \leq n} \rho(n - t + 2) + t - 2$, where $\rho(m)$ is the smallest positive divisor d of m for which $d \geq \sqrt{m}$. This result is verified by computer for all $n < 1765$ and proven for all $n \geq 1765$. For $n = 5, 12, 21$, or 32 , the partitions of n can be reconstructed from their k -minors if and only if k is at most 1, 3, 5, or 7, respectively. Together, these results solve this reconstruction problem completely.

Pretzel and Simons [7] demonstrated that partitions of n can be reconstructed from their sets of k -minors if $n \geq 2k^2 + 8k + 6$, and asked whether this bound is the best possible. In fact, it is not: we can improve this result to $n \geq k^2 + 2k$. This bound is the best possible, in the sense that for every k there exist two distinct partitions of $k^2 + 2k - 1$ which have the same set of k -minors.

The problem of partition reconstruction arises naturally in representation theory. The character reconstruction problem for finite groups [7] asks when we can uniquely recover the character of

E-mail address: monks@mit.edu.

a representation of a finite group G over a field of characteristic zero from its restriction to various subgroups. In Section 4.1, we show that our results on partition reconstruction solve this problem when G is the symmetric group S_n acting on $\{1, 2, \dots, n\}$, and the subgroup is the stabilizer of any subset of $\{1, 2, \dots, n\}$.

Reconstruction of partitions also has applications to related reconstruction problems. Define a cycle k -minor of a permutation $p \in S_n$ to be a permutation in S_{n-k} formed by deleting k elements of the decomposition of p into disjoint cycles and re-numbering the remaining entries from 1 to $n - k$, preserving the relative order of the entries. The problem of reconstructing a permutation from its set of cycle k -minors is currently open [6]. In Section 4.2, we demonstrate that for any k , we can reconstruct the conjugacy class of a permutation in S_n from its cycle k -minors for sufficiently large n .

1.1. Notation

We now introduce the definitions needed to state the main results. Further notation will be provided as needed in Section 3.

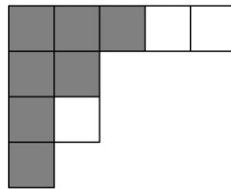
Let n be a positive integer. A *partition* λ of n is an array $[\lambda_1, \lambda_2, \dots, \lambda_m]$ of positive integers which satisfy $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$ and $\sum_{i=1}^m \lambda_i = n$. If $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_m]$ is a partition of n , we say that n is the *size* of λ , and we call $\lambda_1, \lambda_2, \dots, \lambda_m$ the *parts* of λ . For any partition λ , we will always use λ_1 to denote the largest part, λ_2 the second largest, and so on, and we define $\lambda_j = 0$ for any j larger than the number of parts of λ . We now introduce the notion of a minor of a partition.

Definition. Let λ be a partition of n , and let μ be a partition of $n - k$. Then μ is a k -minor of λ if $\mu_i \leq \lambda_i$ for all i . We write $M_k(\lambda)$ to denote the set of all k -minors of λ .

The *Young diagram* of a partition $\lambda = [\lambda_1, \dots, \lambda_m]$ is a partial grid of squares consisting of m rows, aligned at the left, with the i th row containing λ_i squares for each $i \leq m$. Henceforth, we will refer to a partition and its Young diagram interchangeably.

A *corner square* of the Young diagram of a partition λ is a square X for which there are no squares directly below or directly to the right of X . We write λ/X to denote the partition whose Young diagram is formed by removing X from λ . In general, a k -minor of λ can be formed by iterating k times the process of removing a single corner square, beginning with the Young diagram of λ . We also write λ/μ to denote the set of squares in a partition λ that are not in its minor μ .

For example, consider the partition $\lambda = [5, 2, 2, 1]$. The Young diagram of λ is shown below. The Young diagram of its 3-minor $\mu = [3, 2, 1, 1]$ is shaded, and we see that the Young diagram of μ fits inside that of λ .



Continuing with this example, we find that

$$M_3([5, 2, 2, 1]) = \{[5, 2], [5, 1, 1], [4, 2, 1], [4, 1, 1, 1], [3, 2, 2], [3, 2, 1, 1], [2, 2, 2, 1]\}.$$

We wish to find the pairs of integers n and k for which $M_k(\lambda) = M_k(\nu)$ implies that $\lambda = \nu$ for all partitions λ and ν of n . If this property is satisfied for a given n and k , we say *reconstructibility holds*, and otherwise it *fails*. The partition reconstruction problem asks when reconstructibility holds. To answer this, we require the following number theoretic function.

Definition. For any positive integer m , let $\rho(m)$ be the smallest divisor d of m for which $d \geq \sqrt{m}$.

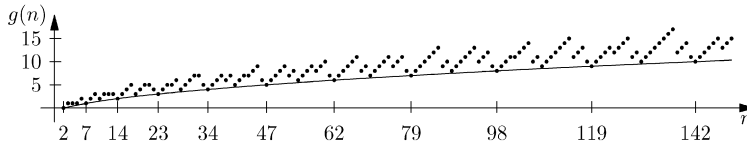


Fig. 1. The plot of $g(n)$ for $2 \leq n \leq 150$, along with the lower bound of $\sqrt{n+2} - 2$. The lower bound is achieved when $n+2$ is a perfect square, as indicated.

2. Main results

In this section we state the main results and defer the proofs until Section 3.

Theorem 2.1. Let n and k be positive integers with $k < n$. For $n \notin \{5, 12, 21, 32\}$, define

$$g(n) = \min_{0 \leq t \leq n} \rho(n+2-t) - 2 + t$$

and also define $g(5) = 1$, $g(12) = 3$, $g(21) = 5$, $g(32) = 7$. Then partitions of n can be reconstructed from their sets of k -minors if and only if $k \leq g(n)$.

Theorem 2.1 provides us with an efficient means of determining whether reconstructibility holds for a given n and k by finding the minimum of a set of only n values. In fact, there are even more efficient ways of computing $g(n)$, as the following two theorems show.

Theorem 2.2. Let $n > 2$ be a positive integer other than 5, 12, 21, and 32. Then

$$g(n) = \min\{\rho(n+2) - 2, g(n-1) + 1\}. \quad (2.1)$$

Furthermore, we have the following explicit formula when n is two less than a square.

Theorem 2.3. Suppose $n+2$ is a perfect square. Then $g(n) = \sqrt{n+2} - 2$.

Theorems 2.2 and 2.3 enable us to compute $g(n)$ without computing all values of $\rho(n+2-t) - 2 + t$ for $0 \leq t \leq n$. Given n , we can first find the largest $m \leq n$ for which $m+2$ is a perfect square, compute $g(m)$ using Theorem 2.3, and then use (2.1) to compute $g(m+1)$, $g(m+2)$, \dots , $g(n)$. For example, $g(63) = \min\{\rho(65) - 2, g(62) + 1\} = \min\{13 - 2, \sqrt{64} - 2 + 1\} = 7$.

Theorem 2.3 can be extended to several other infinite families of positive integers n . For instance, if d is a fixed positive integer and $n = r(r+d) - 2$ for some positive integer r , then as long as r is sufficiently large compared to d we have $g(n) = \rho(n+2) - 2$. The proof of this fact is similar to that of Theorem 2.3, and we omit it.

The order of growth of g is approximately \sqrt{n} , as the following theorem illustrates.

Theorem 2.4. For all positive integers $n \geq 2$,

$$\sqrt{n+2} - 2 \leq g(n) \leq \sqrt{n+2} + 3\sqrt[4]{n+2}.$$

Hence, for large n we can approximate $g(n)$ as $\sqrt{n+2}$. This is usually unnecessary due to the formulas above, but it provides useful intuition about the values of g .

While g is clearly not an invertible function (see Fig. 1), we can provide a tight bound on reconstructibility for a fixed positive integer k .

Theorem 2.5. Let k be a positive integer. Then reconstructibility holds for $n \geq k^2 + 2k$, and fails for $n = k^2 + 2k - 1$.

Notice that there are some values of n that are less than $k^2 + 2k - 1$ for which any partition of size n can be reconstructed from its set of k -minors. For example, for $k = 6$, reconstructibility holds for $n = 27, 30, 31, 32, 36, 37, 38, 39, 41, 42, 43, 44, 45, 46$. It fails for $n = 47$, but for $n \geq 6^2 + 2 \cdot 6 = 48$, partitions of n can always be reconstructed from their 6-minors.

3. Proofs

We first prove Theorem 2.2, which we restate below.

Theorem 2.2. *Let $n > 2$ be a positive integer other than 5, 12, 21, and 32. Then*

$$g(n) = \min\{\rho(n+2) - 2, g(n-1) + 1\}.$$

Proof. It can be verified by a direct calculation that this recursion holds for $n = 6, 13, 22$, and 33 . Suppose $n \notin \{5, 6, 12, 13, 21, 22, 32, 33\}$. Then by the definition of g ,

$$\begin{aligned} g(n) &= \min_{0 \leq t \leq n} \rho(n+2-t) - 2 + t \\ &= \min_{-1 \leq t \leq n-1} \rho(n+2-(t+1)) - 2 + (t+1) \\ &= \min\left\{\rho(n+2) - 2, \left(\min_{0 \leq t \leq n-1} \rho((n-1)+2-t) - 2 + t\right) + 1\right\} \\ &= \min\{\rho(n+2) - 2, g(n-1) + 1\} \end{aligned}$$

as desired. \square

We proceed to prove the lower bound of Theorem 2.4.

Lemma 3.1. *For all positive integers $n \geq 2$,*

$$\sqrt{n+2} - 2 \leq g(n).$$

Proof. A straightforward calculation shows that the inequality holds for $n \leq 32$ (see Fig. 1). Suppose $n \geq 33$. Then by the definition of g , we have $g(n) = \rho(n+2-t) - 2 + t$ for some t such that $0 \leq t \leq n$. Thus

$$\begin{aligned} g(n) &= \rho(n+2-t) - 2 + t \\ &\geq \sqrt{n+2-t} - 2 + t \\ &\geq \sqrt{n+2} - 2 \end{aligned}$$

where the final inequality follows from the fact that $\sqrt{n+2-t} - 2 + t$ is an increasing function of t when $0 \leq t \leq n$. \square

Theorem 2.3 now follows.

Theorem 2.3. *Suppose $n+2$ is a perfect square. Then $g(n) = \sqrt{n+2} - 2$.*

Proof. Suppose $n+2$ is a perfect square. Then $\rho(n+2) - 2 = \sqrt{n+2} - 2$, so $g(n) \leq \sqrt{n+2} - 2$ by the definition of g . Furthermore, from Lemma 3.1 we have $g(n) \geq \sqrt{n+2} - 2$, and hence $g(n) = \sqrt{n+2} - 2$. \square

We now prove the upper bound of Theorem 2.4.

Lemma 3.2. For all positive integers $n \geq 2$,

$$g(n) \leq \sqrt{n+2} + 3\sqrt[4]{n+2}.$$

Proof. Straightforward computation shows that the bound holds for $n \leq 32$. Suppose $n \geq 33$, so that the recursion (2.1) holds. Note that for any positive integers a and r for which $r^2 - a^2 - 2 \geq 2$, we have

$$\begin{aligned} g(r^2 - a^2 - 2) &\leq \rho(r^2 - a^2) - 2 \\ &= \rho((r-a)(r+a)) - 2 \\ &\leq r + a - 2 \end{aligned}$$

by the definitions of g and ρ , and similarly

$$g(r^2 - r - a(a+1) - 2) \leq r + a - 2$$

whenever $r^2 - r - a(a+1) - 2 \geq 2$. Furthermore, iterating the inequality $g(m+1) \leq g(m) + 1$ (a consequence of Theorem 2.2), we have

$$g(m+t) \leq g(m) + t$$

for all $t \geq 0$ and $m \geq 2$. Combining these, we obtain the following two inequalities:

$$g(r^2 - a^2 - 2 + t) \leq r + a - 2 + t, \quad (3.1)$$

$$g(r^2 - r - a(a+1) - 2 + t) \leq r + a - 2 + t. \quad (3.2)$$

Now, let $r = \lceil \sqrt{n+2} \rceil$, so that $(r-1)^2 + 1 \leq n+2 \leq r^2$.

Case 1. Suppose $(r-1)^2 + 1 \leq n+2 \leq r^2 - r - 1$. Let a be the smallest positive integer such that $r^2 - r - a(a+1) - 2 \leq n$. Then

$$n = r^2 - r - a(a+1) - 2 + t$$

for some $t \leq a(a+1) - (a-1)a - 1 = 2a - 1$. In addition, by the definition of a we have $n \leq r^2 - r - (a-1)a - 2 - 1$. Since $(r-1)^2 - 1 \leq n$, we have $(r-1)^2 - 1 \leq r^2 - r - (a-1)a - 2 - 1$, which we can solve for a to obtain

$$a \leq \frac{1}{2} + \sqrt{r - \frac{11}{4}}.$$

Therefore, $t \leq 2a - 1 \leq 2\sqrt{r - \frac{11}{4}}$. By (3.2), we have

$$\begin{aligned} g(n) &= g(r^2 - r - a(a+1) - 2 + t) \\ &\leq r + a - 2 + t \\ &\leq r + 3\sqrt{r - \frac{11}{4}} - \frac{3}{2} \\ &= \lceil \sqrt{n+2} \rceil + 3\sqrt{\lceil \sqrt{n+2} \rceil - \frac{11}{4}} - \frac{3}{2} \\ &\leq \sqrt{n+2} + 3\sqrt[4]{n+2} \end{aligned}$$

as desired.

Case 2. Suppose $r^2 - r \leq n+2 \leq r^2 - 1$. Let a be the smallest positive integer such that $r^2 - a^2 - 2 \leq n$. Then $n = r^2 - a^2 - 2 + t$ for some $t \leq a^2 - (a-1)^2 - 1 = 2a - 2$. In addition, by the definition of a we

have $n \leq r^2 - (a-1)^2 - 3$. Since $r^2 - r - 2 \leq n$ as well, it follows that $r^2 - r - 2 \leq r^2 - (a-1)^2 - 3$, and so

$$a \leq \sqrt{r-1} + 1.$$

Therefore $t \leq 2a - 2 \leq 2\sqrt{r-1}$. By (3.1), we have

$$\begin{aligned} g(n) &= g(r^2 - a^2 - 2 + t) \\ &\leq r + a - 2 + t \\ &\leq r + 3\sqrt{r-1} - 1 \\ &= \lceil \sqrt{n+2} \rceil + 3\sqrt{\lceil \sqrt{n+2} \rceil} - 1 - 1 \\ &\leq \sqrt{n+2} + 3\sqrt[4]{n+2} \end{aligned}$$

as desired.

Case 3. Suppose $n+2 = r^2$. By Theorem 2.3 we have $g(n) = \sqrt{n+2} - 2 \leq \sqrt{n+2} + 3\sqrt[4]{n+2}$.

Hence, $g(n) \leq \sqrt{n+2} + 3\sqrt[4]{n+2}$ for all n . \square

Theorem 2.4, which we restate below, follows directly from Lemmas 3.1 and 3.2.

Theorem 2.4. For all positive integers $n \geq 2$,

$$\sqrt{n+2} - 2 \leq g(n) \leq \sqrt{n+2} + 3\sqrt[4]{n+2}.$$

To prove the remaining theorems, we first introduce some new terminology.

Definition. Let λ and μ be any two partitions. The *union* of λ and μ is the partition $\lambda \cup \mu$ whose i th part is $\max\{\lambda_i, \mu_i\}$ for all i . Similarly, the *intersection* of λ and μ is the partition $\lambda \cap \mu$ whose i th part is $\min\{\lambda_i, \mu_i\}$ for all i .

In other words, the union or intersection of two partitions is formed by taking the union or intersection, respectively, of the sets of squares in their Young diagrams.

Definition. Let X be a square of the Young diagram of a partition λ . Then the *outer region* of X , denoted $\text{Out}_\lambda(X)$, is the set of all squares that lie strictly below or strictly to the right of X , and the *inner region* of X , denoted $\text{In}_\lambda(X)$, is the rectangle of squares with corners at X and the upper left-hand corner of the diagram (see Fig. 2).

We often refer to the outer region or inner region of X simply as $\text{Out}(X)$ or $\text{In}(X)$ when the partition in question is clear. Notice that $\text{In}_\lambda(X)$ is always a minor of λ , and $\lambda / \text{In}_\lambda(X) = \text{Out}_\lambda(X)$.

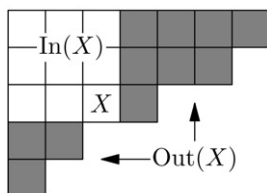


Fig. 2. A square X , with its outer region shaded.

Lemma 3.3. *Let λ and ν be partitions of n and let $\kappa = \lambda \cup \nu$. Then $M_k(\lambda) = M_k(\nu)$ if and only if $|\text{Out}_\nu(X)| \leq k - 1$ for all squares $X \in \kappa/\lambda$ and $|\text{Out}_\lambda(Y)| \leq k - 1$ for all $Y \in \kappa/\nu$.*

Proof. Suppose $M_k(\lambda) = M_k(\nu)$, and assume there exists a square $X \in \kappa/\lambda$ having $|\text{Out}_\nu(X)| \geq k$. Then there exists a k -minor μ of ν that contains X , formed by removing k corner squares in succession from $\text{Out}_\lambda(X)$. Since X is not in λ , the minor $\mu \in M_k(\nu)$ cannot be in $M_k(\lambda)$, which is a contradiction since $M_k(\lambda) = M_k(\mu)$. Similarly, if there exists a square $Y \in \kappa/\nu$ with $|\text{Out}_\lambda(Y)| \geq k$, then there is a minor of λ that is not a minor of ν . Hence, $|\text{Out}_\nu(X)| \leq k - 1$ for all squares $X \in \kappa/\lambda$ and $|\text{Out}_\lambda(Y)| \leq k - 1$ for all $Y \in \kappa/\nu$.

Conversely, suppose $|\text{Out}_\nu(X)| \leq k - 1$ for all $X \in \kappa/\lambda$ and $|\text{Out}_\lambda(Y)| \leq k - 1$ for all $Y \in \kappa/\nu$. Let μ be a k -minor of λ , and let $X \in \kappa/\nu$ be arbitrary. Assume that μ contains the square X . Then μ contains $\text{In}(X)$, and since $|\text{In}(X)| + |\text{Out}_\lambda(X)| = n$ we have $|\text{In}(X)| > n - k$. Hence, μ contains more than $n - k$ squares, a contradiction. Since X was arbitrary, it follows that μ cannot contain any square in κ/ν , and so μ is a k -minor of ν as well. By a similar argument, any k -minor μ of ν is a minor of λ , and so $M_k(\lambda) = M_k(\nu)$. \square

We now introduce a metric on partitions.

Definition. Let λ and ν be any two partitions. Then the *distance* between λ and ν , denoted $d(\lambda, \nu)$, is given by

$$\sum_{i=1}^{\infty} |\lambda_i - \nu_i|.$$

Alternatively, the distance between λ and ν is the number of squares that appear in the Young diagram of either λ or ν but not in both. This yields the identity

$$d(\lambda, \nu) = |\lambda \cup \nu| - |\lambda \cap \nu|. \quad (3.3)$$

Notice that if λ and ν are partitions of the same size and $d(\lambda, \nu) = 2$, then $\lambda \cup \nu$ has exactly one corner square that is in ν but not in λ and exactly one corner square that is in λ but not in ν . Thus we obtain the following corollary to Lemma 3.3.

Corollary 3.4. *Let λ and ν be any two partitions of n having $d(\lambda, \nu) = 2$. Let X and Y be the unique corner squares of $\lambda \cup \nu$ that are not in λ and ν , respectively. Then $M_k(\lambda) = M_k(\nu)$ if and only if each of $|\text{Out}_\nu(X)|$ and $|\text{Out}_\lambda(Y)|$ is at most $k - 1$.*

We now show that if reconstructibility fails for n and k , there are two partitions λ and ν of n with $d(\lambda, \nu) = 2$ that have the same set of k -minors.

Lemma 3.5. *Let k be a positive integer, and suppose n is a positive integer for which there are partitions $\lambda \neq \mu$ of n for which $M_k(\lambda) = M_k(\mu)$. Then there exists a partition ν of n such that $d(\lambda, \nu) = 2$ and $M_k(\lambda) = M_k(\nu)$.*

Proof. First note that since $\lambda \neq \mu$ we have $d(\lambda, \mu) > 0$. Also, by (3.3) and the inclusion–exclusion principle, we see that $d(\lambda, \mu) = |\lambda \cup \mu| - |\lambda \cap \mu| = |\lambda| + |\mu| - 2|\lambda \cap \mu| = 2n - 2|\lambda \cap \mu|$ is even, so $d(\lambda, \mu) \neq 1$. Hence $d(\lambda, \mu) \geq 2$.

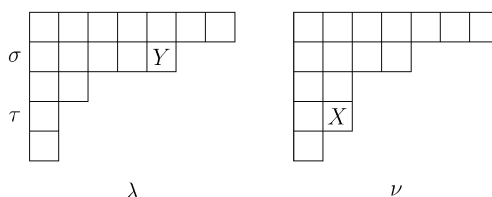
We now construct ν as follows. The sizes of λ and μ are equal and $d(\lambda, \mu) \geq 2$, so there must exist indices s and t such that $\mu_s < \lambda_s$ and $\lambda_t < \mu_t$. We can assume without loss of generality that $s < t$. Hence, $\mu_t \leq \mu_s$ and so $\lambda_t + 2 \leq \lambda_s$. Let σ be the largest index such that $\lambda_\sigma = \lambda_s$, and let τ be the smallest index such that $\lambda_\tau = \lambda_t$. Notice that we have defined σ and τ so that row λ_σ contains a corner square, and adding a square to row λ_τ will result in a new partition. We also have $\mu_\sigma < \lambda_\sigma$ and $\lambda_\tau < \mu_\tau$. Since $\lambda_\tau + 2 \leq \lambda_\sigma$, it follows that we can move a square in the Young diagram from the part λ_σ to λ_τ to form a new partition.

Let m be the number of parts of λ , and let ν be the partition

$$[\lambda_1, \lambda_2, \dots, \lambda_\sigma - 1, \dots, \lambda_\tau + 1, \dots, \lambda_m]$$

formed by moving a square of the Young diagram from the part λ_σ to λ_τ . Notice that $d(\lambda, \nu) = 2$. We proceed to show that $M_k(\lambda) = M_k(\nu)$.

Let X be the square in row τ that is in ν but not in λ , and let Y be the square in row σ that is in λ but not in ν as shown below. By Corollary 3.4, it suffices to show that each of $|\text{Out}_\nu(X)|$ and $|\text{Out}_\lambda(Y)|$ is at most $k - 1$. Notice that Y is not in μ since $\mu_\sigma < \lambda_\sigma$. Since $M_k(\lambda) = M_k(\mu)$, it follows from Lemma 3.3 that $|\text{Out}_\lambda(Y)| \leq k - 1$. Also, since $\lambda_\tau < \mu_\tau$, the square X is in μ but not in λ , so $|\text{Out}_\mu(X)| \leq k - 1$. Furthermore, X is in the same row and column in μ as it is in ν , so $|\text{In}_\mu(X)| = |\text{In}_\nu(X)|$. Since μ and ν have the same size n , it follows that $|\text{Out}_\nu(X)| = |\text{Out}_\mu(X)| \leq k - 1$ as desired. \square



We now provide a necessary and sufficient condition for reconstructibility to hold for a given n and k . We write $c \bmod a$ to denote the remainder when c is divided by a .

Lemma 3.6. *Let n and k be positive integers. Then there exist partitions $\lambda \neq \mu$ of n with $M_k(\lambda) = M_k(\mu)$ if and only if n can be expressed in the form*

$$n = (a + 1)b + c - 1$$

for some positive integers a , b , and c satisfying $a \leq c \leq k$ and $b + (c \bmod a) \leq k$.

Proof. First, suppose $n = (a + 1)b + c - 1$ for some positive integers a , b , and c satisfying $a \leq c \leq k$ and $b + (c \bmod a) \leq k$. Let $r = c \bmod a$ and $q = (c - r)/a$ so that $c = aq + r$. Consider the partition $\kappa = [a + 1, a + 1, \dots, a + 1, a, a, \dots, a, r]$ that contains b parts equal to $a + 1$, q parts equal to a , and one part equal to r . Then κ is a partition of $n + 1$. Let λ be the 1-minor of κ formed by removing the corner square X that appears in the last part in κ equal to $a + 1$, namely, κ_b , and let μ be the partition formed by removing the corner square Y appearing in the last part equal to a , namely, κ_{b+q} . Then $|\text{Out}_\kappa(X)| = aq + r = c \leq k$ and $|\text{Out}_\kappa(Y)| = b + r = b + (c \bmod a) \leq k$. Hence, each of $|\text{Out}_\mu(X)|$ and $|\text{Out}_\lambda(Y)|$ is at most $k - 1$, so by Corollary 3.4 we have $M_k(\lambda) = M_k(\mu)$.

Conversely, suppose n and k are such that there exist partitions $\lambda \neq \mu$ of n with $M_k(\lambda) = M_k(\mu)$. We will show that we can find a partition κ' of $n + 1$ having two squares X' and Y' that are either in adjacent rows or adjacent columns in the Young diagram of κ' , and such that $|\text{In}(X')| \geq n - k$ and $|\text{In}(Y')| \geq n - k$.

By Lemma 3.5, there exists a partition ν of n such that $d(\lambda, \nu) = 2$ and $M_k(\lambda) = M_k(\nu)$. Let $\kappa = \lambda \cup \nu$. Let X be the square in κ that lies outside of λ and let Y be the square in κ that lies outside of ν . Assume without loss of generality that X lies above and to the right of Y . Suppose X is in row b and Y is in column a , so that $\text{In}(X) \cap \text{In}(Y)$ is an $a \times b$ rectangle of squares. Let c and d be such that Y is in row $b + c$ and X is in column $a + d$. We may also assume without loss of generality that $a \leq b$, by interchanging the rows and columns if necessary.

If $c = 1$ or $d = 1$, then we can set $\kappa' = \kappa$. Otherwise, let c' be the largest positive integer such that $(a + 1)c' \leq ac$. By this definition, we have $(a + 1)(c' + 1) > ac$, so

$$(a + 1)c' \geq ac - a. \tag{3.4}$$

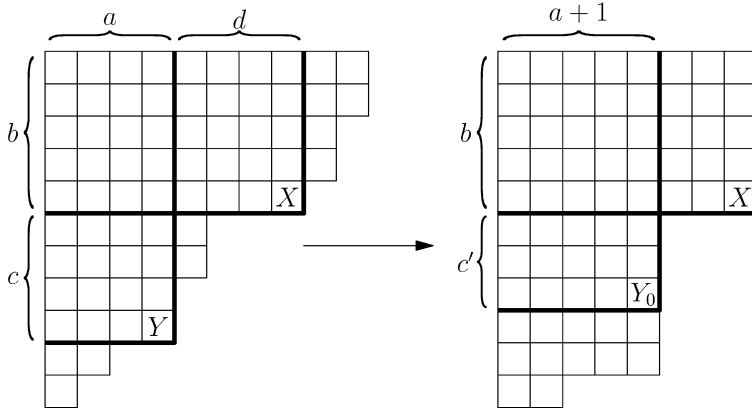


Fig. 3. Moving Y closer to X as in the proof of Lemma 3.6.

Let $m = |\text{Out}_\kappa(X)|$, and write $m = (a+1)q + r$ where q and r are nonnegative integers with $0 \leq r \leq a$. Define η to be the partition having b parts equal to $a+d$, followed by q parts equal to $a+1$, and one part equal to r . In other words, we stack all of the squares in the outer region of X in rows of $a+1$ (with r left over) below row b , as in Fig. 3. Let Y_0 be the last square in row $b+c'$ in η . Notice that Y_0 is closer to X than Y is, both vertically and horizontally.

We clearly have $|\text{In}_\eta(X)| = |\text{In}_\kappa(X)|$, which is at least $n-k$ by Lemma 3.3. Using (3.4) and our assumption that $b \geq a$, we have

$$\begin{aligned} |\text{In}_\eta(Y_0)| &= (a+1)(b+c') \\ &= (a+1)b + (a+1)c' \\ &\geq (a+1)b + ac - a \\ &= a(b+c) + b - a \\ &= |\text{In}_\kappa(Y)| + b - a \\ &\geq |\text{In}_\kappa(Y)|. \end{aligned}$$

Since $|\text{In}_\kappa(Y)| \geq n-k$ by Lemma 3.3, we have $|\text{In}_\eta(Y_0)| \geq n-k$.

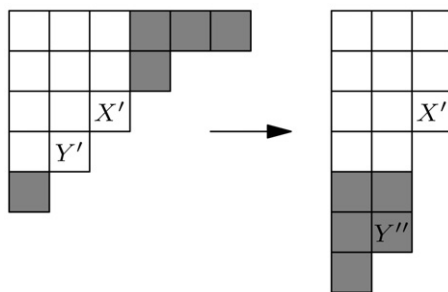
We can now continue this process starting with η and new values a and b formed by the intersection of the inner regions of Y_0 and X . Hence, we can form a partition κ' having X' and Y' either in adjacent rows or adjacent columns and each of $|\text{In}_{\kappa'}(X')|$ and $|\text{In}_{\kappa'}(Y')|$ is at least $n-k$. This implies that each of $|\text{Out}_{\kappa'}(X')|$ and $|\text{Out}_{\kappa'}(Y')|$ is at most k .

Finally, consider such a partition κ' . Suppose, without loss of generality (by interchanging rows and columns if necessary), that X' and Y' are in adjacent columns with Y' below and to the left of X' , as in Fig. 4. Let p be the number of squares of the Young diagram that are in both of $\text{Out}(X')$ and $\text{Out}(Y')$. Suppose Y' is in column a of the Young diagram. Write $p = aq + r$ where q and r are nonnegative integers with $0 \leq r < a$. Consider the partition κ'' formed by removing all p aforementioned squares from κ' , and then adding q rows of a and one row of r squares below the row containing Y' . Let Y'' be the new corner square in column a . Then κ'' has the form $[a+1, a+1, \dots, a+1, a, a, \dots, a, r]$. Let b be the number of parts of κ'' equal to $a+1$, and let s be the number of parts equal to a . Define $c = as + r$. Then we see that $n+1 = (a+1)b + c$. Also, $a \leq c = as + r = p + a(s-q) = |\text{Out}_{\kappa'}(X')| \leq k$, and $b + (c \bmod a) = b + r \leq b + p = |\text{Out}_{\kappa'}(Y')| \leq k$.

Therefore, there exist partitions $\lambda \neq \mu$ of n with $M_k(\lambda) = M_k(\mu)$ if and only if n can be expressed in the form

$$n = (a+1)b + c - 1$$

for some positive integers a , b , and c satisfying $a \leq c \leq k$ and $b + (c \bmod a) \leq k$. \square

Fig. 4. Creating κ'' .

Theorem 2.5 follows immediately, and we provide the proof below.

Theorem 2.5. *Let k be a positive integer. Then reconstructibility of partitions of n from their k -minors holds for $n \geq k^2 + 2k$, and fails for $n = k^2 + 2k - 1$.*

Proof. Let n and k be positive integers that satisfy the conditions in Lemma 3.6 for some a , b , and c . Then the inequality $b + (c \bmod a) \leq k$ implies $b \leq k$, so each of a , b , and c are at most k . Hence, for a given k , the largest value of n for which reconstructibility fails occurs when $a = b = c = k$ and $n = k^2 + 2k - 1$. \square

We now introduce an auxiliary function, h , which we will later show is identical to g .

Definition. Let n be a positive integer. Then $h(n)$ is the largest value k for which partitions of n can be reconstructed from their k -minors for all $k \leq h(n)$.

By this definition, partitions of n cannot be reconstructed from their $(h(n) + 1)$ -minors. We proceed to show that in fact there are no values k larger than $h(n) + 1$ for which partitions of n can be reconstructed from their k -minors.

Lemma 3.7. *Reconstructibility of partitions of n from their k -minors holds if and only if $k \leq h(n)$.*

Proof. By the definition of h , the smallest positive integer k for which partitions of n cannot be reconstructed from their k -minors is $h(n) + 1$. Let $k_0 = h(n) + 1$. Then there exist partitions λ and ν of n such that $M_{k_0}(\lambda) = M_{k_0}(\nu)$. Let $k \geq k_0$ be arbitrary. Then $M_k(\lambda)$ consists of all $(k - k_0)$ -minors of the elements of $M_{k_0}(\lambda)$, and similarly $M_k(\nu)$ consists of the $(k - k_0)$ -minors of the elements of $M_{k_0}(\nu)$, so $M_k(\lambda) = M_k(\nu)$. Hence, reconstructibility fails for all $k \geq h(n) + 1$, and the claim follows. \square

To prove Theorem 2.1, it now suffices to show that $h(n) = g(n)$ for all n . We first prove an intermediate lemma, which provides a formula for h .

Lemma 3.8. *Let n be a positive integer, and let S be the set of all solutions (a, b, s, t) to the Diophantine equation $n = (a + 1)b + sa + t - 1$ for which a, b, s are positive integers and t is nonnegative. Then*

$$h(n) = \min_{(a,b,s,t) \in S} (\max\{sa + t, b + t\}) - 1.$$

Proof. Let n be a positive integer. Recall that $h(n) + 1$ is the smallest value k for which there are two partitions of n having the same set of k -minors. By Lemma 3.6, this is the smallest value k for which there exists a solution (a, b, c) in positive integers to $n = (a + 1)b + c - 1$ satisfying $a \leq c \leq k$ and $b + (c \bmod a) \leq k$.

Let R be the set of all solutions (a, b, c) in positive integers to $n = (a + 1)b + c - 1$ for which $a \leq c$. Let $(a_1, b_1, c_1) \in R$. The smallest k_1 for which $c_1 \leq k_1$ and $b_1 + (c_1 \bmod a_1) \leq k_1$ is $k_1 = \max\{c_1, b_1 + (c_1 \bmod a_1)\}$. Hence, the smallest value k for which two partitions of n have the same set of k -minors is $\min_{(a,b,c) \in R} (\max\{c, b + (c \bmod a)\})$. It follows that $h(n) = \min_{(a,b,c) \in R} (\max\{c, b + (c \bmod a)\}) - 1$, and so it suffices to show that $\min_{(a,b,c) \in R} (\max\{c, b + (c \bmod a)\}) = \min_{(a,b,s,t) \in S} (\max\{sa + t, b + t\})$.

Let $m = \min_{(a,b,c) \in R} (\max\{c, b + (c \bmod a)\})$ and let $(a_0, b_0, c_0) \in R$ such that $\max\{c_0, b_0 + (c_0 \bmod a_0)\} = m$. Since $a_0 \leq c_0$ by the definition of R , we can write $c_0 = s_0 a_0 + t_0$ for some positive integer s_0 and nonnegative integer $t_0 < a_0$. Then $(a_0, b_0, s_0, t_0) \in S$. Furthermore, $t_0 = c_0 \bmod a_0$, so $\max\{s_0 a_0 + t_0, b_0 + t_0\} = \max\{c_0, b_0 + (c_0 \bmod a_0)\} = m$. Hence m is attained as a value of $\max\{sa + t, b + t\}$ for some $(a, b, s, t) \in S$.

Finally, assume that there exists a solution $(a, b, s, t) \in S$ such that $\max\{sa + t, b + t\} < m$. Write $sa + t = s'a + t'$ where s' is a positive integer and $0 \leq t' < a$. Then $t' \leq t$, so $b + t' \leq b + t$. It follows that $\max\{s'a + t', b + t'\} \leq \max\{sa + t, b + t\} < m$. Let $c = s'a + t'$. Then $t' = c \bmod a$ and $a \leq c$, so $(a, b, c) \in R$ satisfies $\max\{c, b + (c \bmod a)\} = \max\{s'a + t', b + t'\} < m$. This is a contradiction since m is the minimum possible value of $\max\{s'a + t', b + t'\}$. Hence $m = \min_{(a,b,s,t) \in S} (\max\{sa + t, b + t\})$ as desired. \square

We finally have the tools to prove our main result.

Theorem 2.1. Let n and k be positive integers with $k < n$. For any positive integer $n \notin \{5, 12, 21, 32\}$, define

$$g(n) = \min_{0 \leq t \leq n} \rho(n + 2 - t) - 2 + t$$

and also define $g(5) = 1$, $g(12) = 3$, $g(21) = 5$, $g(32) = 7$. Then partitions of n can be reconstructed from their sets of k -minors if and only if $k \leq g(n)$.

Proof. We wish to show that $g(n) = h(n)$ for all $n \geq 2$. By a straightforward calculation, this holds for $n = 2, 3, 5, 9, 12, 21$, and 32 .

For a fixed positive integer N , consider the Diophantine equation

$$N = (a + 1)b + sa + t - 1 \tag{3.5}$$

in positive integers a, b, s and nonnegative integers t . Define a *minimal solution* to this equation to be a solution (a, b, s, t) for which the value $\max\{sa + t, b + t\} - 1$ attains its minimum. Notice that if $N = n$ and $s = 1$, we have $n - t + 2 = (a + 1)(b + 1)$. Hence, for any nonnegative integer t we have $\min(\max\{sa + t, b + t\}) - 1 = \min(\max\{a + 1, b + 1\}) - 2 + t = \rho(n + 2 - t) - 2 + t$, where the minimum is taken over all a and b satisfying $n - t + 2 = (a + 1)(b + 1)$. Taking the smallest such value over all nonnegative integers t , we see that if there exists a minimal solution to (3.5) having $s = 1$ then the minimum value is $g(n)$, and so $h(n) = g(n)$ by Lemma 3.8. Hence, it suffices to show that a minimal solution to (3.5) having s as small as possible has $s = 1$.

To do so, we use induction on n . As base cases, it is easily verified that this property holds for $n = 4, 6, 10, 13, 22$, and 33 .

Now, let $n \notin \{1, 2, 3, 4, 5, 6, 9, 10, 12, 13, 21, 22, 32, 33\}$ be an arbitrary positive integer and assume that one of the minimal solutions to (3.5) for $N = n - 1$ having s as small as possible has $s = 1$, and hence that $h(n - 1) = g(n - 1)$. Let (a, b, s, t) be a minimal solution to (3.5) for $N = n$ having s as small as possible. We show that $s = 1$.

First, suppose $t \geq 1$. Then $(a, b, s, t - 1)$ is a solution to (3.5) when $N = n - 1$. We claim that this must be a minimal solution for $N = n - 1$. For, assume that (a', b', s', t') is a solution for $N = n - 1$ satisfying $\max\{s'a' + t', b' + t'\} < \max\{sa + t - 1, b + t - 1\}$. Then $(a', b', s', t' + 1)$ is a solution for $N = n$ and $\max\{s'a' + t' + 1, b' + t' + 1\} < \max\{sa + t, b + t\}$, which is impossible since we assumed that (a, b, s, t) was minimal for $N = n$. Thus, since $(a, b, s, t - 1)$ is a minimal solution for $N = n - 1$, we have $s = 1$ by the inductive hypothesis.

Now, suppose $t = 0$, so that $n = ab + b + sa - 1$, and assume to the contrary that $s \geq 2$. We consider several cases.

Case 1. Suppose $sa + 2 \leq b$. Then $a \leq (b - 2)/s$ and $n = (a + 1)b + sa - 1$, so we have

$$\begin{aligned} n &\leq ((b - 2)/s + 1)b + s(b - 2)/s - 1, \\ sn &\leq (b - 2 + s)b + sb - 2s - s, \\ (s - 1)^2 + sn + 3s &\leq b^2 + 2(s - 1)b + (s - 1)^2, \\ \sqrt{sn + s^2 + s + 1} &\leq b + (s - 1), \\ 1 - s + \sqrt{sn + s^2 + s + 1} &\leq b. \end{aligned}$$

It is straightforward to verify that for a fixed n , the expression $1 - s + \sqrt{sn + s^2 + s + 1}$ is an increasing function of s for $s > 0$, so using our assumption that $s \geq 2$ we have $1 - 2 + \sqrt{2n + 2^2 + 2 + 1} \leq b$. Hence

$$b \geq \sqrt{2n + 7} - 1.$$

Since $sa + 2 \leq b$, we have $h(n) = \max\{sa, b\} = b$. Notice that if (a, b, s, t) is a solution to (3.5) for $N = n - 1$, then $(a, b, s, t + 1)$ is a solution to (3.5) for $N = n$, and so by Lemma 3.8 we have $h(n) \leq h(n - 1) + 1$. Thus, to obtain a contradiction it suffices to show that $b \geq h(n - 1) + 2$. Notice that the function $\sqrt{2n + 7} - 1$ has a larger order of growth than $\sqrt{n + 1} + 3\sqrt[4]{n + 1} + 2$. The inequality $\sqrt{2n + 7} - 1 \geq \sqrt{n + 1} + 3\sqrt[4]{n + 1} + 2$ holds for all $n \geq 4360$. It follows that for $n \geq 4360$, we have $b \geq \sqrt{n + 1} + 3\sqrt[4]{n + 1} + 2 \geq g(n - 1) + 2 = h(n - 1) + 2$ by Theorem 2.4 and the inductive hypothesis. Hence $b > h(n)$, a contradiction.

For $n < 4360$, a computer calculation shows that in fact $g(n - 1) \leq \sqrt{n + 1} + 3\sqrt[4]{n + 1} - 5$. In addition, the inequality $\sqrt{2n + 7} - 1 \geq \sqrt{n + 1} + 3\sqrt[4]{n + 1} - 3$ holds for all $n \geq 1765$. By a similar argument to that above, we now have that if $n \geq 1765$ then $b > h(n)$.

For $n < 1765$, a computer calculation verifies that no minimal solution has $sa + 2 \leq b$ and $s \geq 2$.

In each of the remaining cases, we will show that there is a solution (a', b', s', t') to (3.5) having $\max\{s'a' + t', b' + t'\} \leq \max\{sa, b\}$ and $1 \leq s' < s$, so that we can decrease s and still form a minimal solution, thereby obtaining a contradiction.

Case 2. Suppose $a + 2 \leq b \leq sa + 1$. Then we can rewrite n as follows:

$$\begin{aligned} n &= ab + b + sa - 1 \\ &= (a + 1)(b - 1) + sa + a \\ &= (a + 1)(b - 1) + (b - 1) + (sa + a - b + 2) - 1 \\ &= a'b' + b' + s'a' + t' - 1 \end{aligned}$$

where $a' = a + 1$, $b' = b - 1$, and $s'a' + t' = sa + a - b + 2$ such that $0 \leq t' < a'$.

To show that $1 \leq s'$, note that since $b \leq sa + 1$ we have $a + 1 \leq sa + a - b + 2 = s'(a + 1) + t'$. Furthermore, since $a + 2 \leq b$, we have $s'a' \leq s'a' + t' = sa + (a + 2) - b \leq sa < sa'$, so $s' < s$.

Finally, we have $s'a' + t' \leq sa$ and also $b' + t' = (b - 1) + (s'a' + t') - s'a' = (b - 1) + (sa + a - b + 2) - s'(a + 1) \leq (b - 1) + (sa + a - b + 2) - (a + 1) = sa$, so $\max\{s'a' + t', b' + t'\} \leq sa \leq \max\{sa, b\}$ as desired.

Case 3. Suppose $b \leq a + 1$ and $s \geq 3$. We write

$$\begin{aligned} n &= ab + b + sa - 1 \\ &= a(b + 1) + (b + 1) + (sa - a - 1) - 1 \\ &= a'b' + b' + s'a' + t' - 1 \end{aligned}$$

where $a' = a$, $b' = b + 1$, and $s'a' + t' = sa - a - 1$ such that $0 \leq t' < a'$.

To show $1 \leq s'$, note that since $s \geq 3$ and $a \geq 1$ we have $a \leq 3a - (a + 1) \leq sa - a - 1 = s'a + t'$. Furthermore, since $s'a' + t' < sa'$, we must have $s' < s$.

Finally, we have $s'a' + t' \leq sa' = sa$ and also $b' + t' \leq b + 1 + a - 1 = b + a \leq 2a + 1 \leq 3a \leq sa$, so $\max\{s'a' + t', b' + t'\} \leq sa \leq \max\{sa, b\}$ as desired.

Case 4. Suppose $b \leq a - 1$ and $s = 2$. We write

$$\begin{aligned} n &= ab + b + sa - 1 \\ &= (a + 1)b + b + (2a - b) - 1 \\ &= a'b' + b' + s'a' + t' - 1 \end{aligned}$$

where $a' = a + 1$, $b' = b$, and $s'a' + t' = 2a - b$ such that $0 \leq t' < a'$.

To show $1 \leq s'$, note that since $b \leq a - 1$ we have $a + 1 \leq 2a - b = s'a' + t' = s'(a + 1) + t'$. Furthermore, since $s'(a + 1) + t' = 2a - b < 2(a + 1)$ we have $s' \leq 1$. Hence $s' = 1$ (and therefore $1 \leq s' < s$).

Finally, we have $s'a' + t' < 2a = sa$ and also $b' + t' \leq a - 1 + a < 2a = sa$, so $\max\{s'a' + t', b' + t'\} \leq sa \leq \max\{sa, b\}$ as desired.

In the remaining two cases, we finally discover the significance of the seemingly mysterious values 3, 5, 9, 12, 21, and 32.

Case 5. Suppose $b = a$ and $s = 2$. Then

$$\begin{aligned} n &= ab + b + sa - 1 \\ &= a^2 + 3a - 1 \\ &= (a + 2)(a - 1) + 2a + 1 \\ &= (a + 2)(a - 1) + (a - 1) + ((a + 2) + 1) - 1 \\ &= a'b' + b' + s'a' + t' - 1 \end{aligned}$$

where $a' = a + 2$, $b' = a - 1$, and $s' = t' = 1$. Clearly $1 \leq s' < s$.

Notice that $b' + t' = a < 2a = sa$, and $s'a' + t' = a + 3 \leq 2a$ if and only if $a \geq 3$, so when $a \geq 3$ we have $\max\{s'a' + t', b' + t'\} \leq sa \leq \max\{sa, b\}$. If $a = 1$ or $a = 2$, we obtain the extraneous values $n = 3$ and $n = 9$.

Case 6. Suppose $b = a + 1$ and $s = 2$. Then

$$\begin{aligned} n &= ab + b + sa - 1 \\ &= a^2 + 4a \\ &= (a - 1)(a + 3) + 2a + 3 \\ &= (a - 1)(a + 3) + (a + 3) + (a + 1) - 1 \\ &= a'b' + b' + s'a' + t' - 1 \end{aligned}$$

where $a' = a - 1$, $b' = a + 3$, $s' = 1$, and $t' = 2$. For $a \geq 5$, we have $s'a' + t' = a + 1 \leq 2a = sa$ and $b' + t' \leq a + 3 + 2 \leq 2a = sa$, and so $\max\{s'a' + t', b' + t'\} \leq sa \leq \max\{sa, b\}$. For $a = 1, 2, 3$, and 4 , we obtain the extraneous values $n = 5, 12, 21$, and 32 , for which every minimal solution to (3.5) has $s \geq 2$.

This completes the proof. \square

4. Applications and future work

In this section, we present two direct applications of our results and propose a natural extension of the partition reconstruction problem to Young tableaux.

4.1. The character reconstruction problem for S_n

Suppose G is a finite group and \mathcal{H} is a collection of subgroups of G . For any representation x of G over a field of characteristic zero, define $\text{Irr}(x)$ to be the set of irreducible representations appearing as composition factors in the decomposition of x into irreducible representations. Similarly, if χ is the character corresponding to x , define $\text{Irr}(\chi)$ to be the set of irreducible characters corresponding to the elements of $\text{Irr}(x)$. The equivalence relation $\sim_{\mathcal{H}}$ on the irreducible representations of G is defined by $x \sim_{\mathcal{H}} y$ if and only if $\text{Irr}(x|_H) = \text{Irr}(y|_H)$ for all $H \in \mathcal{H}$, where $x|_H$ denotes the restriction of x to H . The equivalence $\chi \sim_{\mathcal{H}} \phi$ is defined in a similar manner for irreducible characters χ and ϕ of G .

The character reconstruction problem for finite groups is stated in [7] as follows. For which collections \mathcal{H} does $\chi \sim_{\mathcal{H}} \phi$ imply that $\chi = \phi$ for any two irreducible characters χ and ϕ of G ?

Consider the symmetric group S_n , the group of permutations of $\{1, 2, \dots, n\}$. It is well known that there is a one-to-one correspondence between irreducible representations of S_n and partitions of n . (See [4,5], or [10] for a more detailed discussion of the representation theory of the symmetric groups.) There is a natural way to construct this correspondence such that if H is the stabilizer of some k -element subset of $\{1, 2, \dots, n\}$, then the representation x^λ associated with a partition λ satisfies $\text{Irr}(x^\lambda|_H) = \{x^\mu : \mu \in M_k(\lambda)\}$. This is known as the Branching Theorem.

Now, suppose \mathcal{H} consists of a single subgroup, H , which stabilizes some k points in $\{1, 2, \dots, n\}$. It follows from the Branching Theorem that an irreducible representation x^λ of S_n can be recovered from its restriction to H if and only if λ can be reconstructed from its set of k -minors. This argument holds for characters as well as representations, so we obtain the following corollary to Theorem 2.1.

Corollary 4.1. *Let n and k be positive integers with $k < n$, and let $H \subset S_n$ be the stabilizer of a k -element subset of $\{1, 2, \dots, n\}$. Then any irreducible representation x of S_n (and hence its character χ) can be reconstructed from the set of irreducible composition factors of $x|_H$ if and only if $k \leq g(n)$.*

4.2. An application to permutation reconstruction

A k -reduction of a permutation $p = p_1 p_2 \dots p_n$ of $\{1, 2, \dots, n\}$ is a permutation of $\{1, 2, \dots, n - k\}$ formed by re-numbering the elements of an $(n - k)$ -element subsequence of $p_1 p_2 \dots p_n$ such that the relative order of the elements is preserved. For instance, 132 is a 2-reduction of the permutation $p = 31524$ by considering the subsequence 154 of p . The problem of reconstructing permutations from certain sets or multisets of k -reductions has been of much recent interest [1–3,8,9].

A natural variant on this problem that is less well understood is the reconstruction of permutations from their cycle k -minors. Given a permutation written as a product of disjoint cycles, a cycle k -minor is formed by removing some k of the elements and re-numbering the remaining elements so as to preserve their order. For example, $(315)(24)$ is a cycle 1-minor of $(4162)(35)$, formed by deleting the 2 from the cycle (4162) , and then subtracting 1 from every remaining number that is larger than 2. It has been shown [6] that all permutations in S_n can be reconstructed from their sets of cycle 1-minors if and only if $n \geq 6$, and conjectures that for any positive integer k , we can reconstruct permutations in S_n from their cycle k -minors for sufficiently large n .

Theorem 2.1 provides an interesting insight into this problem. Recall that the conjugacy classes of S_n consist of all permutations which are a product of disjoint cycles having a given number of cycles of each length. Hence we can associate each partition λ of n with the conjugacy class in S_n consisting of the permutations p having one λ_i -cycle for each i in the decomposition of p into disjoint cycles. For example, the partition $[3, 3, 2]$ is associated with the permutations in S_8 having disjoint cycle decomposition of the form $(abc)(def)(gh)$.

Clearly, the partition associated with a cycle 1-minor of a permutation p is a 1-minor of the partition associated with p . Thus we have the following corollary to Theorem 2.1.

Corollary 4.2. *The conjugacy class of a permutation can be reconstructed from its set of cycle k -minors whenever $k \leq g(n)$.*

In the case $k = 1$, this is not sufficient to reconstruct the permutation as well, since reconstructibility holds for $n \geq 3$ for partitions, whereas $n \geq 6$ is required to reconstruct permutations from their cycle 1-minors. Nevertheless, this may be a useful intermediate step in solving this conjecture.

4.3. Reconstructing Young tableaux

Having solved the partition reconstruction problem, it would be interesting to extend this question to Young tableaux, which also arise naturally in representation theory. A (standard) *Young tableau* of size n is a Young diagram of a partition of n whose squares are labeled with the numbers $1, 2, \dots, n$ such that the labels are increasing from left to right in each row and from top to bottom in each column.

We propose a natural definition of a minor of a Young tableau inspired by *jeu de taquin*, or “the teasing game.” (See [10, pp. 419–425].) Suppose we remove a square X and its label from a Young tableau and re-number the squares from 1 to $n - 1$, again preserving the relative order of the labels. If X was a corner square, we are left with a tableau of size $n - 1$. Otherwise, consider the square Y directly to the right of X and the square Z below X (note that either of Y or Z may not exist). If Y has a smaller label than Z or Z does not exist, slide Y to the left, and otherwise slide Z up to fill in the missing square. Continue this sliding process until a new tableau is formed. We define this to be a 1-minor of the tableau, and similarly define a k -minor to be a tableau formed by taking k successive 1-minors.

Theorem 2.1 shows that we can reconstruct the *shape* of the tableau from its set of k -minors whenever $k \leq g(n)$, since every possible k -minor of the corresponding partition will appear as the shape of some k -minor formed by removing corner squares in succession. However, this is not always sufficient to reconstruct the labeling of the squares. For example, the two tableaux of size 4 shown below have the same set of 1-minors. This prompts the question of which n and k have the property that any tableau with n squares can be reconstructed from its set of k -minors.

1	3
2	4

1	2
3	4

5. Addendum

Since submitting this paper for publication it has come to our attention that Vatter [11] also independently showed that reconstructibility holds for n and k holds whenever $n \geq k^2 + 2k$. This is essentially equivalent to Lemma 3.1.

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References

- [1] S. Avgustinovich, S. Kitaev, On uniquely k -determined permutations, *Discrete Math.* 308 (2008) 1500–1507.
- [2] J. Ginsburg, Determining a permutation from its set of reductions, *Ars Combin.* 82 (2007).

- [3] M. Ince, Permutation reconstruction from the set of minors, preprint, 2006.
- [4] G. James, *The Representation Theory of the Symmetric Groups*, Lecture Notes in Math., Springer-Verlag, Berlin, 1978.
- [5] A. Kleshchev, Branching rules for symmetric groups and applications, in: *Algebraic Groups and Their Representations*, Cambridge Univ. Press, 1997, pp. 103–130.
- [6] M. Monks, Reconstructing permutations from their cycle minors, preprint, 2007.
- [7] O. Pretzel, J. Siemons, Reconstruction of partitions, *Electron. J. Combin.* 13 (2005), Note 5, 6.
- [8] M. Raykova, Permutation reconstruction from minors, *Electron. J. Combin.* 13 (2006), Research paper 66, 14.
- [9] R. Smith, Permutation reconstruction, *Electron. J. Combin.* 13 (2006), N11.
- [10] R.P. Stanley, *Enumerative Combinatorics*, vol. 2, Cambridge Univ. Press, Cambridge, 1999.
- [11] V. Vatter, A sharp bound for the reconstruction of partitions, *Electron. J. Combin.* 15 (2008), N23.